PERGAMON

International Journal of Solids and Structures 37 (2000) 1743–1763

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

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Rectangular thick plates on winkler foundation: differential quadrature element solution

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Received 16 June 1998; in revised form 2 October 1998

Abstract

This paper deals with the static analysis of homogenous isotropic rectangular plates on Winkler foundation on the basis of first-order shear deformation theory. An improved differential quadrature (DQ) method, called the differential quadrature element method (DQEM), has been developed for this analysis. The plates considered are subjected to a patch load or a concentrated line load, which are not solvable by the global DQ method. The convergence and comparison studies are carried out to establish the reliability of the DQEM results. Then the numerical results for different boundary conditions (i.e. SSSS, CCCC, S′S′S′S′ and SFSF) are presented showing the parametric effects of dimensions of loading area/line, relative thickness ratio and elastic foundation modulus on the deflection, bending and twisting moments, and shear forces at selected locations. Most of these data are new and due to the high accuracy of the DQ solution they can be useful for benchmarking future work. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Among the various thick plate theories available today, the most widely used thick plate theory is the first-order shear deformation theory, which was first proposed by Reissner (1945), and developed further for dynamic problems by Mindlin (1951). A number of analytical and numerical methods for the rectangular thick plates resting on an elastic foundation have been reported based on Reissner's plate theory (Frederick, 1957; Voyiadjis and Baluch, 1979; Svec, 1976) and Mindlin's plate theory (Kobayashi and Sonoda, 1989; Liew et al., 1996). Frederick (1957) presented an analytical solution using the Navier-type and Levy-type series method. Voyiadjis and Baluch (1979) proposed an approximate approach for solving a simply supported rectangular plate on Winkler foundation. Kobayashi and Sonoda (1989) presented a Levy-type solution for the rectangular plates on Winkler foundations with two opposite

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PII: S0020-7683(98)00306-0

simply supported edges, and two other edges being arbitrarily restrained. Svec (1976) studied the similar problem by the finite element method. Malik et al. (1993) obtained the solution of uniformly loaded circular plates resting on elastic half space using the differential quadrature method. Liew et al. (1996) presented a differential quadrature solution to the rectangular Mindlin plates on Winkler foundation with arbitrary combinations of boundary conditions. Moreover, the static analysis of rectangular plates with elastic foundation based on other refined plate theories can also be found in literature (Henwood et al., 1981, 1982; Yettram et al., 1984; Voyiadjis and Kattan, 1986).

All the above solutions, however, are restricted to plates subjected only to a uniformly distributed load over the entire plate surface. Solutions to the thick plates with elastic foundations and subjected to a patch load or a concentrated line load are scare in the open literature. Voyiadjis and Kattan (1990) derived the governing equations for non-symmetrical bending of thick plates on elastic foundations based on a refined plate theory. An infinite thick plate subjected to line and concentrated loads was analyzed. However, this analytical solution is only applicable to very special cases. Even in the case of the uniform loading on the entire surface of plate, the solutions are only possible for plates with two opposite edges simply supported. Canisius and Foschi (1993) treated the similar problem by using the finite strip method. Feng and Owen (1996) developed an iterative scheme for the coupled FE/BE analysis of a plate–foundation interaction problem. A rectangular plate which freely rests on an elastic half-space foundation and is subjected to a point load is examined. This iterative scheme can be done within an integrated FEM/BEM software environment formed by merging the FEM and BEM programs. However, the overall performance of the scheme depends largely on the choice of a free parameter α and a matrix K_L contained in the scheme. The approach on how to determine the optimal value of α was not given in the paper. No other solutions have been found on the thick plates on Winkler foundations involving the patch loading or concentrated loading. In particular, the results presented by Liew et al. (1996) have shown that the differential quadrature method is highly efficient and accurate for solving the bending problems of thick rectangular plates on Winkler foundation. This numerical method was originated by Bellman and Casti (1972, 1973), and thanks to the efforts of Bert et al. (1988, 1989), Striz et al. (1988) and Sherbourne and Pandey (1991), it is becoming increasingly popular in the structural mechanics field (Liew and Liu, 1998; Liu and Liew, 1998a–b, 1999a–c). A notable review paper on both the theoretical development and the application of the DQ method has been published by Bert and Malik (1996). Unfortunately, however, this numerical method, by its very basis, is only applicable to problems which should satisfy the continuity conditions (Striz et al., 1994; Chen et al., 1997a, b; Bert and Malik, 1996). It can not be employed directly to solve the thick plates on elastic foundation subjected to a patch load or other discontinuous loading conditions. Striz, Chen and Bert developed the quadrature element method (QEM) to solve the bending of truss and beam (Striz et al., 1994) and free vibration of thin plate (Chen et al., 1997a, b) having discontinuities and obtained excellent solutions for these problems, but no solution has been provided for thick plates with discontinuities. Han and Liew (1996) developed an improved technique, called the differential quadrature element method (DQEM) for one dimensional bending analysis of the axisymmetric circular Mindlin plate. Wang and Gu (1997) also developed one dimensional differential quadrature element method (DQEM) for the thick beam analysis. Recently, Liu and Liew developed the two-dimensional DQEM for static analyses of rectangular thick plates (Liu and Liew, 1998b) and discontinuous polar plates (Liu and Liew, 1999b). Further, the method has been developed for the solution of free vibration problems of the discontinuous shear deformable plates (Liu and Liew, 1999c). The primary objective of this paper is to develop this methodology for solving the bending problems of thick rectangular plates on Winkler foundation. The domain decomposition technique is introduced to combine with the DQ method for this analysis. The methodology developed here is different from the QEM since it is based on different plate theory with different governing equations and only one grid point is employed to represent the interface point between elements in which no δ-grid arrangement is needed. The static responses of the moderately thick rectangular plates on Winkler foundation and subjected to a patch load or a concentrated line load are carefully investigated for different boundary conditions to reveal the effects of the dimensions of loading area/line, plate thickness and elastic foundation modulus on the solution values.

2. Mathematical formulations

Consider a rectangular thick plate resting on a Winker foundation with side lengths $a \times b$. The plate is divided into N_E elements based on the discontinuities in the geometry, boundary constraints and materials used. Each element consists of an isotropic material, has uniform thickness and continuous boundary constraints on each edge and is subjected to a continuously distributed load. For the lth element, the thickness of the plate, Young's modulus, shear modulus, Poisson's ratio, and the load intensity are represented by h_l, E_l, G_l, v_l and q_l respectively.

2.1. Basic equations

For a given element l as shown in Fig. 1, the equilibrium equations of a thick plate on Winkler foundation are given in terms of the displacement components and based on the first-order shear deformation plate theory as follows (Kobayashi and Sonoda, 1989):

$$\frac{D_l}{2}\left[(1-v_l)\nabla^2\psi_x + (1+v_l)\frac{\partial\phi}{\partial x}\right] - \kappa G_l h_l\left(\frac{\partial w}{\partial x} + \psi_x\right) = 0 \tag{1a}$$

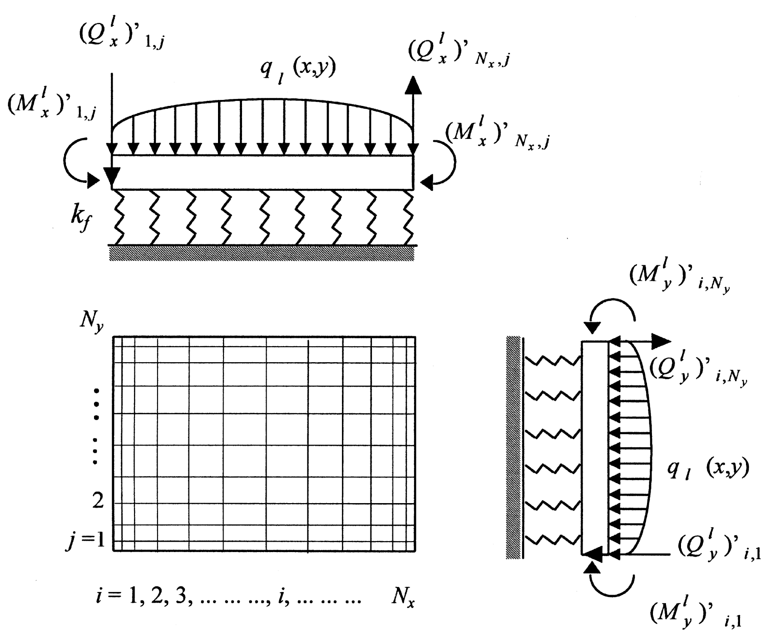

Fig. 1. Arrangement of grid points for element l on elastic foundation.

$$\frac{D_l}{2}\left[(1-\nu_l)\nabla^2\psi_y+(1+\nu_l)\,\frac{\partial\phi}{\partial y}\right]-\kappa G_l h_l\left(\frac{\partial w}{\partial y}+\psi_y\right)=0 \tag{1b}$$

$$\kappa G_l h_l\left(\nabla^2 w+\phi\right)+q_l-\kappa_f w=0 \tag{1c}$$

in which

$$\phi=\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y} \tag{2}$$

$$\nabla^2=\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2} \tag{3}$$

$$D_l=\frac{E_l h_l^3}{12\left(1-\nu_l^2\right)} \tag{4}$$

w is the transverse deflection; ψ_x and ψ_y are the rotations of the normal about the y-axis and x-axis respectively; k_f is the elastic foundation modulus; D_l is the plate flexural rigidity and $\kappa\,(=5/6)$ is the shear correction factor.

The moments and shear forces are expressed as

$$M_x=D_l\left(\frac{\partial\psi_x}{\partial x}+\nu_l\frac{\partial\psi_y}{\partial y}\right] \tag{5a}$$

$$M_y=D_l\left(\nu_l\frac{\partial\psi_x}{\partial x}+\frac{\partial\psi_y}{\partial y}\right] \tag{5b}$$

$$M_{xy}=\frac{1-\nu_l}{2}D_l\left(\frac{\partial\psi_x}{\partial y}+\frac{\partial\psi_y}{\partial x}\right] \tag{5c}$$

$$Q_x=\kappa G_l h_l\left(\frac{\partial w}{\partial x}+\psi_x\right) \tag{5d}$$

$$Q_y=\kappa G_l h_l\left(\frac{\partial w}{\partial y}+\psi_y\right) \tag{5e}$$

The boundary conditions for the sides of a rectangular plate can be divided into four kinds. Taking side $x=0$ for example, the boundary conditions are expressed as

1. Generalized hard simply supported sides (S):

$$w=0,\quad \psi_y=0,\quad M_x=\left(M_x^{\text{ext}}\right)_1; \tag{6}$$

2. Generalized soft simply supported sides (S′):

$$w=0,\quad M_{xy}=0,\quad M_x=\left(M_x^{\text{ext}}\right)_1; \tag{7}$$

3. Clamped sides (C):

$$w = 0, \quad \psi_x = 0, \quad \psi_y = 0; \tag{8}$$

4. Generalized free sides (F):

$$Q_x = \left(Q_x^{\text{ext}}\right)_1, \quad M_x = \left(M_x^{\text{ext}}\right)_1, \quad M_{xy} = 0 \tag{9}$$

where $(M_x^{\text{ext}})_1$ and $(Q_x^{\text{ext}})_1$ are the concentrated external line moments and loads at the side $x = 0$.

2.2. Rectangular DQEM plate element on elastic foundation

Further dividing the lth element into $N_x \times N_y$ grid points along the x- and y-axis respectively and applying the DQM rule (Liew et al., 1996), the equilibrium eqns (1a)–(1c) can be discretized at each discrete point on the inner grid of the element l as

$$\sum_{k=1}^{N_x} C_{ik}^{(2)}\left(\psi_x\right)_{kj} + \frac{(1-\nu_l)}{2}\sum_{m=1}^{N_y}\bar{C}_{jm}^{(2)}\left(\psi_x\right)_{im} - F_l\left(\psi_x\right)_{ij}$$

$$+\frac{(1+\nu_l)}{2}\left[\sum_{k=1}^{N_x} C_{ik}^{(1)}\sum_{m=1}^{N_y}\bar{C}_{jm}^{(1)}\left(\psi_y\right)_{km}\right] - F_l\sum_{k=1}^{N_x} C_{ik}^{(1)}(w)_{kj} = 0 \tag{10a}$$

$$\frac{(1+\nu_l)}{2}\left[\sum_{k=1}^{N_x} C_{ik}^{(1)}\sum_{m=1}^{N_y}\bar{C}_{jm}^{(1)}\left(\psi_y\right)_{km}\right] + \sum_{m=1}^{N_y}\bar{C}_{jm}^{(2)}\left(\psi_y\right)_{im}$$

$$+\frac{(1-\nu_l)}{2}\sum_{k=1}^{N_x} C_{ik}^{(2)}\left(\psi_y\right)_{kj} - F_l\left(\psi_y\right)_{ij} - F_l\sum_{m=1}^{N_y}\bar{C}_{jm}^{(1)}(w)_{im} = 0 \tag{10b}$$

$$\sum_{k=1}^{N_y} C_{ik}^{(2)}(w)_{kj} + \sum_{m=1}^{N_y}\bar{C}_{jm}^{(2)}(w)_{im} + \sum_{k=1}^{N_x} C_{ik}^{(1)}\left(\psi_x\right)_{kj} + \sum_{m=1}^{N_y}\bar{C}_{jm}^{(1)}\left(\psi_y\right)_{im} - \frac{k_f}{\kappa G_l h_l} w_{ij} = -\frac{q_l}{\kappa G_l h_l}$$

$$i = 1, 2, 3, \ldots, N_x; \quad j = 1, 2, 3, \ldots, N_y; \quad N_y, \quad l = 1, 2, 3, \ldots, N_E \tag{10c}$$

where $F_l = 6\kappa(1-\nu_l^2)/h_l^2$; $C_{rs}^{(n)}$ and $\bar{C}_{rs}^{(n)}(r = 1, 2, 3, \ldots, N_x;\ s = 1, 2, 3, \ldots, N_y)$ are the weighting coefficients for the nth-order partial derivatives of w, ψ_x and ψ_y with respect to the global coordinates x and y.

At the four edges of element l, the governing eqns (10a)–(10c) should be replaced by the boundary conditions or compatibility conditions. If the edge is located at the sides of the plate, the boundary conditions (6)–(9) are used, otherwise, the compatibility conditions should be employed.

The matrix form of eqns (10a)–(10c) can be written as

$$\mathbf{K}^{\mathrm{e}}\mathbf{d}^{\mathrm{e}} = \mathbf{f}^{\mathrm{e}} \tag{11}$$

in which $\mathbf{K}^{\mathrm{e}}$, $\mathbf{d}^{\mathrm{e}}$ and $\mathbf{f}^{\mathrm{e}}$ are defined as the element weighting coefficient matrix, element displacement vector and element force vector, respectively, and

$$\mathbf{d}^{e} = \left[w_{1,1}, \psi_{x1,1}, \psi_{y1,1}, w_{1,2}, \psi_{x1,2}, \psi_{y1,2}, \ldots, w_{N_x,N_y}, \psi_{xN_x,N_y}, \psi_{yN_x,N_y}\right]^T \tag{12}$$

$$\mathbf{f}^{e} = \Big[\big(Q_x^l\big)'_{1,1}, \big(M_x^l\big)'_{1,1}, \big(M_{xy}^{l}{}'_{1,1}\big), \big(Q_x^l\big)'_{1,2}, \big(M_x^l\big)'_{1,2}, \big(M_{xy}'^{l}\big)'_{1,2}, \ldots, \big(Q_x^l\big)'_{1,Ny},$$

$$\big(M_x^l\big)'_{1,N_y}, \big(M_{xy}^l\big)'_{1,N_y}, \big(Q_y^l\big)'_{2,1}, \big(M_y^l\big)'_{2,1}, \big(M_{xy}^l\big)'_{2,1}, 0, 0, f_{2,2},$$

$$0, 0, f_{2,3}, \ldots, 0, 0, f_{2,N_y-1}, \big(Q_y^l\big)'_{2,N_y}, \big(M_y^l\big)'_{2,N_y}, \big(M_{xy}^l\big)'_{2,N_y}, \ldots,$$

$$\big(Q_x^l\big)'_{N_x,1}, \big(M_x^l\big)'_{N_x,1}, \big(M_{xy}^l\big)'_{N_x,1}, \big(Q_x^l\big)'_{N_x,2}, \big(M_x^l\big)'_{N_x,2}, \big(M_{xy}^l\big)'_{N_x,2}, \ldots, \big(Q_x^l\big)'_{N_x,N^y}, \big(M_x^l\big)'_{N_x,N_y}, \big(M_{xy}^l\big)'_{N_x,N_y}\Big]^T \tag{13}$$

where

$$f_{ij} = \frac{q_{ij}^l}{\kappa G_l h_l}, \quad i = 2, 3, \ldots, N_x - 1; \quad j = 2, 3, \ldots, N_y - 1. \tag{14}$$

$(Q_x^l)'_{1,j}$, $(M_x^l)'_{1,j}$, $(M_{xy}^l)'_{1,j}$, $(Q_x^l)'_{N_xj}$, $(M_x^l)'_{N_xj}$, $(M_x^l)'_{N_xj}$, $(Q_y^l)'_{i,1}$, $(M_y^l)'_{i,1}$, $(Q_y^l)'_{i,N_y}$ and $(M_y^l)'_{i,N_y}$ ($i = 1, 2, \ldots, N_x$; $j = 1, 2, \ldots, N_y$) are the combinations of the external forces and moments applied at the four edges of element l, and the shear forces and moments produced by adjacent elements. The expressions of these forces and moments are determined by the compatibility conditions given in Section 2.3. The coefficients in $\mathbf{K}^e$ are determined by eqns (10a)–(10c).

2.3. Assembling plate elements and compatibility conditions

An overall system of equations for all the nodal points of the plate labeled 1 to N should be constructed first in order to obtain a complete solution for the whole plate. This can be simply accomplished by assembling all the element weighting coefficient matrices, force and moment vectors and displacement vectors. The final global matrix form of equation for the whole plate becomes

$$\mathbf{Kd} = \mathbf{F} \tag{15}$$

where $\mathbf{K}$, $\mathbf{d}$ and $\mathbf{F}$ represent the overall weighting coefficient matrix, global displacement vector and global force and moment vector, respectively. The vector $\mathbf{d}$ is expressed as

$$\mathbf{d} = \Big[w_1, (\psi_x)_1, (\psi_y)_1, w_2, (\psi_x)_2, (\psi_y)_2, \ldots, w_N, (\psi_x)_N, (\psi_y)_N\Big]^T \tag{16}$$

Obviously, at the interface boundaries of the elements, the displacement compatibility condition is automatically satisfied since the same global nodal number is used for each conjunction node. Only the equilibrium condition is needed to form the compatibility conditions between the interface boundaries of the DQEM plate elements. Hence, according to the locations of conjunction nodes and the number of the elements meeting at these nodes, the compatibility conditions are expressed as follows:

1. For nodes at which two elements meet

Suppose elements l_1 and l_2 are two adjacent elements as shown in Fig. 2(a) and (b). The compatibility conditions for the conjunction nodes at the interface edge of element l_1 and l_2 connected in the x-direction can be written according to the equilibrium condition as

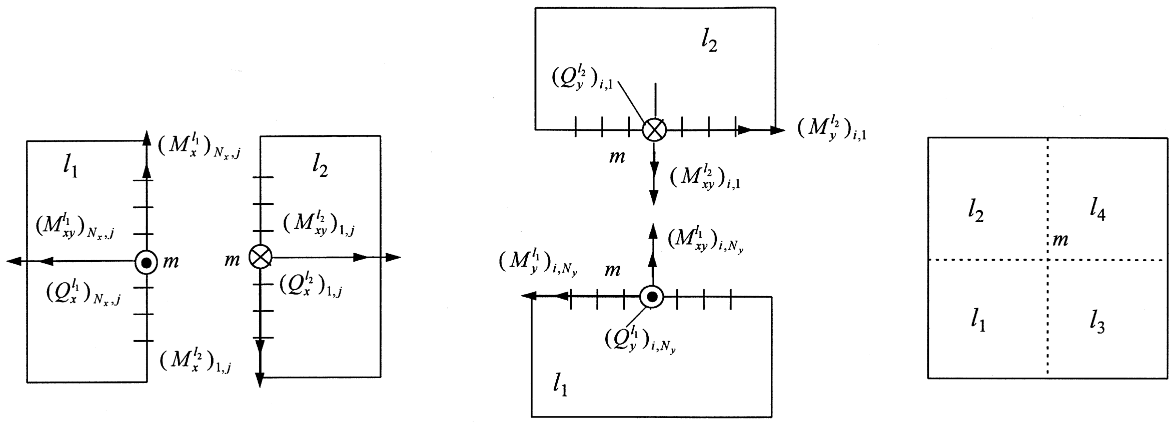

Fig. 2. Locations of the conjunction nodes on the interface boundaries of elements: (*a*) two elements are connected along *x*-axis; (*b*) two elements are connected along *y*-axis; (*c*) four elements are connected at point *m*.

$$\left(Q_x^{l_1}\right)_{N_x,j} - \left(Q_x^{l_2}\right)_{1,j} = \left(Q_x^{\mathrm{ext}}\right)_m \tag{17a}$$

$$\left(M_x^{l_1}\right)_{N_xj} - \left(M_x^{l_2}\right)_{1,j} = \left(M_x^{\mathrm{ext}}\right)_m \tag{17b}$$

$$\left(M_{xy}^{l_1}\right)_{N_xj} - \left(M_{xy}^{l_2}\right)_{1,j} = \left(M_{xy}^{\mathrm{ext}}\right)_m \tag{17c}$$

The compatibility conditions for the conjunction nodes of elements l_1 and l_2 connected in y direction can be obtained similarly as

$$\left(Q_y^{l_1}\right)_{i,N_y} - \left(Q_y^{l_2}\right)_{i,1} = \left(Q_y^{\mathrm{ext}}\right)_m \tag{18a}$$

$$\left(M_y^{l_1}\right)_{i,N_y} - \left(M_y^{l_2}\right)_{i,1} = \left(M_y^{\mathrm{ext}}\right)_m \tag{18b}$$

$$\left(M_{xy}^{l_1}\right)_{i,N_y} - \left(M_{xy}^{l_2}\right)_{i,1} = \left(M_{xy}^{\mathrm{ext}}\right)_m \tag{18c}$$

2. For nodes at which four elements meet

The compatibility conditions for the common node m of the four arbitrarily selected elements, l_1, l_2, l_3 and l_4 as shown in Fig. 2(c) can be expressed as

$$\left(Q_x^{l_1}\right)_{N_x,N_y} + \left(Q_x^{l_2}\right)_{N_x,1} - \left(Q_x^{l_3}\right)_{1,N_y} - \left(Q_x^{l_4}\right)_{1,1} = \left(Q_x^{\mathrm{ext}}\right)_m \tag{19a}$$

$$\left(M_x^{l_1}\right)_{N_x,N_y} + \left(M_x^{l_2}\right)_{N_x,1} - \left(M_x^{l_3}\right)_{1,N_y} - \left(M_x^{l_4}\right)_{1,1} = \left(M_x^{\mathrm{ext}}\right)_m \tag{19b}$$

$$\left(M_{xy}^{l_1}\right)_{N_x,N_y} + \left(M_{xy}^{l_2}\right)_{N_x,1} - \left(M_{xy}^{l_3}\right)_{1,N_y} - \left(M_{xy}^{l_4}\right)_{1,1} = \left(M_{xy}^{\text{ext}}\right)_m \tag{19c}$$

or expressed in terms of the y-components of force/moments at node m of the four elements in the similar way.

3. For conjunction nodes located at the boundaries of plate

For the conjunction nodes located at the side boundary of the plate, both the boundary conditions and the connection conditions should be considered. Take the side boundary $x = 0$ for example. The following modified boundary conditions should be used:

- for clamped edge:

$$w_m = 0, \quad \psi_{xm} = 0, \quad \psi_{ym} = 0 \tag{20}$$

- for hard simply supported edge:

$$w_m = 0, \quad \psi_{ym} = 0, \quad \left(M_x^{l_1}\right)_{1,N_y} + \left(M_x^{l_2}\right)_{1,1} = \left(M_x^{\text{ext}}\right)_m \tag{21}$$

- for soft simply supported edge:

$$w_m = 0, \quad \left(M_x^{l_1}\right)_{1,N_y} + \left(M_x^{l_2}\right)_{1,1} = \left(M_x^{\text{ext}}\right)_m, \quad \left(M_{xy}^{l_1}\right)_{1,N_y} + \left(M_{xy}^{l_2}\right)_{1,1} = \left(M_{xy}^{\text{ext}}\right)_m \tag{22}$$

- for free edge:

$$\left(Q_x^{l_1}\right)_{1,N_y} + \left(Q_x^{l_2}\right)_{1,1} = \left(Q_x^{\text{ext}}\right)_m, \quad \left(M_x^{l_1}\right)_{1,N_y} + \left(M_x^{l_2}\right)_{1,1} = \left(M_{xy}^{\text{ext}}\right)_m$$

$$\left(M_{xy}^{l_1}\right)_{1,N_y} + \left(M_{xy}^{l_2}\right)_{1,1} = \left(M_{xy}^{\text{ext}}\right)_m \tag{23}$$

3. Numerical results and discussion

By assembling all the element weighting coefficient matrices, displacement and force vectors, and considering all the compatibility and boundary conditions, a linear algebraic equation system is obtained. It is solved using the standard linear equation system solver. The grid points are designated as follows:

$$x_i = \frac{a}{2}\left\{1 - \cos\left[(i-1)\pi/(N_x - 1)\right]\right\}; \quad i = 1, 2, 3, \ldots, N_x \tag{24}$$

$$y_j = \frac{b}{2}\left\{1 - \cos\left[(j-1)\pi/\left(N_y - 1\right)\right]\right\}; \quad j = 1, 2, 3, \ldots, N_y \tag{25}$$

The solution procedures developed here can be employed to solve a variety of thick rectangular plate problems with discontinuities in loading, geometry, material and boundary conditions. However, in the following studies, the attention is only paid to the rectangular plates with an elastic (Winkler)

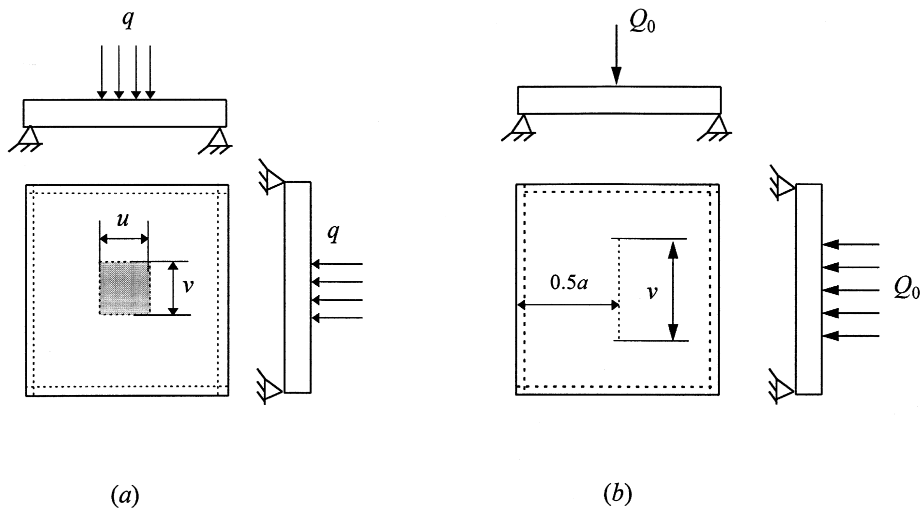

Fig. 3. Loading cases considered for a simply supported rectangular plate: (*a*) subjected to a patch load; (*b*) subjected to a concentrated line load.

foundation subject to two loading cases, i.e. a patch load and a concentrated line load. An illustration of these two loading conditions has been given in Fig. 3(a) and (b) for a simply supported plate. The number of the elements used in computations is 3×3 for the patch loading and 3×2 for the concentrated line loading respectively. The boundary conditions considered are SSSS, CCCC, S′S′S′S′ and SFSF. The notation, for instance, SFSF denotes a rectangular plate with edges $x = 0$, $y = 0$, $x = a$ and $y = b$ having simply supported, free, simply supported and free boundary conditions, respectively.

3.1. Convergence and comparison studies

To examine the validity and accuracy of the DQEM in solution of the problems considered in this paper, the convergence and comparison studies have been carried out first. Table 1 shows the convergence properties of the DQEM solution to a simply supported square plate on a Winkler foundation under a patch load and a concentrated line load for different relative thicknesses. It is evident that the rapid convergence can be obtained for both the thin ($h/a = 0.01$) and the thick ($h/a = 0.20$) plates. No significant effects of the h/a ratio on the convergence rate of the DQEM results for this plate foundation problem have been found. The convergence of the DQEM results for a square plate on Winkler foundations with different boundary conditions under a patch load is presented in Table 2. The grid points in each element are varying from 5×5 to 15×15. In Table 3, the convergence of the DQEM results for the same plate subjected to a concentrated line load is illustrated. It is observed from Tables 2 and 3 that in both loading cases, the DQEM solutions converged very fast with the increasing number of the grid points in each element for all the boundary conditions considered, namely the SSSS, CCCC, S′S′S′S′ and SFSF. Generally, for the patch loading, 7×7 grid points in each element are able to produce a converged solution with at least 3 significant digits. When 9×9 grid points are used, a solution converged to at least 4 significant digits can be obtained for all the boundary conditions considered in Table 2. For the concentrated line loading, the convergence rate is slightly slower than the one for the patch loading. A converged result to at least 3 significant digits can be obtained by using 9×9 grid points in each element. Also observed is that the convergence rates for different boundary conditions are slightly different. However, for all cases considered here, a very

Table 1
The effects of plate relative thickness ratio on convergence of the DQEM results

Loading	h/a	Grid points[a]	$w^{(1)}$ $x/a=0.5$ $y/b=0.5$	$M_x^{(2)}$ $x/a=0.5$ $y/b=0.5$	$M_y^{(2)}$ $x/a=0.5$ $y/b=0.5$	$M_{xy}^{(2)}$ $x/a=0.0$ $y/b=0.0$	$Q_x^{(3)}$ $x/a=0.0$ $y/b=0.5$	$Q_y^{(3)}$ $x/a=0.5$ $y/b=0.0$
Patch load	0.01	5 × 5	1.72234	2.44889	2.44889	−1.0505	0.08196	0.08196
($u/a=v/b=0.5$)		7 × 7	1.77432	2.48078	2.48078	−1.08827	0.07993	0.07993
		9 × 9	1.77460	2.48107	2.48107	−1.08839	0.07992	0.07992
		11 × 11	1.77461	2.48110	2.48110	−1.08839	0.07991	0.07991
		13 × 13	1.77461	2.48110	2.48110	−1.08839	0.07991	0.07991
		15 × 15	1.77461	2.48110	2.48110	−1.08839	0.07991	0.07991
	0.20	5 × 5	2.12217	2.40795	2.40795	−1.04108	0.07636	0.07636
		7 × 7	2.12997	2.39537	2.39537	−1.04502	0.07617	0.07617
		9 × 9	2.13004	2.39565	2.39565	−1.040502	0.07614	0.07614
		11 × 11	2.13004	2.39570	2.39570	−1.04502	0.07614	0.07614
		13 × 13	2.13004	2.39570	2.39570	−1.04502	0.07614	0.07614
		15 × 15	2.13004	2.39570	2.39570	−1.04502	0.07614	0.07614
Line load	0.01	5 × 5	3.73422	8.13628	6.20155	−2.04932	0.00000	0.18546
($v/b=0.5$)		7 × 7	4.12001	9.33740	6.83322	−2.28902	0.00000	0.12818
		9 × 9	4.12833	9.36985	6.86977	−2.28500	0.00000	0.13446
		11 × 11	4.12844	9.35412	6.85777	−2.28516	0.00000	0.13373
		13 × 13	4.12838	9.36044	6.86234	−2.28521	0.00000	0.13393
		15 × 15	4.12838	9.36044	6.86234	−2.28521	0.00000	0.13393
	0.20	5 × 5	5.28248	8.91742	6.60018	−2.14041	0.00000	0.13325
		7 × 7	5.36898	9.14957	6.65354	−2.18868	0.00000	0.12646
		9 × 9	5.37051	9.15376	6.67003	−2.19023	0.00000	0.12646
		11 × 11	5.37045	9.15332	6.66265	−2.19047	0.00000	0.12626
		13 × 13	5.37047	9.15306	6.66614	−2.19053	0.00000	0.12627
		15 × 15	5.37047	9.15306	6.66614	−2.19053	0.00000	0.12627

Results are for a simply supported square plate resting on a Winkler foundation under a patch load and a concentrated line load ($K = 3.0$, $v = 0.3$).
For patch loading: (1) $qa^4 \times 10^{-3}/D$, (2) $qa^2 \times 10^{-2}$, (3) qa; and for concentrated line loading: (1) $Q_0a^3 \times 10^{-3}/D$; (2) $Q_0a \times 10^{-2}$; (3) Q_0.
[a] Grid points in each element.

satisfactory solution with the maximum 0.0036% discrepancy from the completely converged results can be provided using 11 × 11 grid points in each element. Therefore, 11 × 11 grid points will be used for each element to produce all the numerical solutions in the following studies. To examine the accuracy of the present solutions, a comparison study has been given in Table 4 for cases in which the exact solutions are available. Since no exact solutions have been found in the open literature for rectangular plate with a foundation subjected to a patch load or a concentrated line load, the comparisons are only conducted for some special cases (e.g., the uniformly loaded plate with Winkler foundation, and the plate without the foundation but subjected to a patch load or a concentrated line load). The solutions for these cases, of course, are computed using the same program. Excellent agreement is achieved between present results and the exact solutions for all cases tabulated in Table 4. The reliability of present solutions has therefore been confirmed.

3.2. Parametric studies

Based on the convergence and comparison studies above, the deflection, moments and shear forces of a square plate with Winkler foundation and subjected to a patch load and a concentrated line load are

Table 2
Convergence of the DQEM results for a square plate with different boundary conditions subjected to a patch load and resting on Winkler foundation

Boundary conditions	Grid points[a]	$w^{(1)}$ $x/a = 0.5$ $y/b = 0.5$	$M_x^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_y^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_{xy}^{(2)}$ $x/a = 0.0$ $y/b = 0.0$	$Q_x^{(3)}$ $x/a = 0.0$ $y/b = 0.5$	$Q_y^{(3)}$ $x/a = 0.5$ $y/b = 0.0$
SSSS	5 × 5	1.85069	2.46571	2.46571	−1.06761	0.07936	0.07936
	7 × 7	1.86502	2.45903	2.45903	−1.07721	0.07899	0.07899
	9 × 9	1.86510	2.45927	2.45927	−1.07731	0.07895	0.07895
	11 × 11	1.86510	2.45931	2.45931	−1.07731	0.07895	0.07895
	13 × 13	1.86510	2.45932	2.45932	−1.07731	0.07895	0.07895
	15 × 15	1.86510	2.45932	2.45932	−1.07731	0.07895	0.07895
CCCC	5 × 5	0.91741	1.70600	1.70600	0.00000	0.13912	0.13912
	7 × 7	0.92540	1.68740	1.68740	0.00000	0.13733	0.13733
	9 × 9	0.92532	1.68755	1.68755	0.00000	0.13716	0.13716
	11 × 11	0.92531	1.68758	1.68758	0.00000	0.13715	0.13715
	13 × 13	0.92531	1.68758	1.68758	0.00000	0.13715	0.13715
	15 × 15	0.92531	1.68759	1.68759	0.00000	0.13715	0.13715
S′S′S′S′	5 × 5	1.93344	2.52992	2.52984	0.00000	0.11296	0.11317
	7 × 7	1.96513	2.54176	2.54176	0.00000	0.11240	0.11236
	9 × 9	1.96693	2.54344	2.54344	0.00000	0.11233	0.11233
	11 × 11	1.96704	2.54357	2.54357	0.00000	0.11232	0.11232
	13 × 13	1.96704	2.54357	2.54357	0.00000	0.11232	0.11232
	15 × 15	1.96704	2.54358	2.54358	0.00000	0.11232	0.11232
SFSF	5 × 5	2.85175	3.16561	1.80407	−0.00069	0.08010	0.00000
	7 × 7	2.86039	3.15632	1.79864	0.00002	0.07979	0.00000
	9 × 9	2.86045	3.15659	1.79898	0.00000	0.07975	0.00000
	11 × 11	2.86045	3.15664	1.79903	0.00000	0.07975	0.00000
	13 × 13	2.86045	3.15664	1.79904	0.00000	0.07975	0.00000
	15 × 15	2.86045	3.15664	1.79904	0.00000	0.07975	0.00000

(Fig. 1*a*, $K = 3.0$, $v = 3.0$, $h/a = 0.1$, $u/a = v/b = 0.5$).
(1) $qa^4 \times 10^{-3}/D$, (2) $qa^2 \times 10^{-2}$, (3) qa.
[a] Grid points in each element.

determined now for different boundary conditions. The boundary conditions considered here include the SSSS, CCCC, S′S′S′S′ and SFSF. The Poisson's ratio is taken to be $v = 0.3$ for all cases. Table 5 presents the numerical results at several selected locations of the SSSS square plate with different relative thickness ratio h/a, non-dimensional elastic foundation parameter $K(= a^4 k_f/D)$ and dimensions of the loading area $u \times v$. It is observed that the normalized deflection, bending moments at the central point of plate, twisting moment at the plate corner $x = 0$, $y = 0$, and shear forces at the mid-edges of $x = a/2$ and $y = b/2$ increase as the dimensions of the loading area $u \times v$ increase. When the value of $u \times v$ becomes $a \times b$, the solution becomes the one for the uniformly loaded plate. For the fixed values of h/a and $u \times v$, all the normalized deflections, moments and shear forces listed in Table 5 decrease as the non-dimensional elastic foundation parameter K increases from 1.0 to 5.0. The influences of the relative thickness ratio h/a on the numerical results of the plate are also studied in this table. It is found that for the fixed values of K and $u \times v$, the normalized deflection increases when the relative thickness ratio h/a increases from 0.01 to 0.20, whereas the normalized moments and shear forces demonstrate the different responses for different values of K. When $K = 1.0$, as the relative thickness ratio h/a increases from 0.01 to 0.20, all the normalized deflections, moments and shear forces remain nearly unchanged. However, when the value of K increases from 3.0 to 5.0, the values of normalized deflections, moments and shear forces decrease as the relative thickness ratio h/a increases from 0.01 to 0.20. This reflects that

Table 3
Convergence of the DQEM results for a square plate with different boundary conditions subjected to a concentrated load along $x/a = 0.5$ and resting on Winkler foundation

Boundary conditions	Grid points[a]	$w^{(1)}$ $x/a = 0.5$ $y/b = 0.5$	$M_x^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_y^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_{xy}^{(2)}$ $x/a = 0.0$ $y/b = 0.0$	$Q_x^{(3)}$ $x/a = 0.0$ $y/b = 0.5$	$Q_y^{(3)}$ $x/a = 0.0$ $y/b = 0.5$
SSSS	5 × 5	4.25324	8.73081	6.54878	−2.13085	0.00000	0.15403
	7 × 7	4.43953	9.30140	6.79667	−2.25916	0.00000	0.13210
	9 × 9	4.44214	9.30750	6.81607	−2.26073	0.00000	0.13186
	11 × 11	4.44212	9.30625	6.80875	−2.26093	0.00000	0.13195
	13 × 13	4.44213	9.30620	6.81213	−2.26100	0.00000	0.13194
	15 × 15	4.44213	9.30640	6.81035	−2.26102	0.00000	0.13194
CCCC	5 × 5	2.31011	6.95654	4.97835	0.00000	−3.97147	0.26198
	7 × 7	2.42357	7.59865	5.19153	0.00000	−4.46168	0.24209
	9 × 9	2.42335	7.60074	5.20495	0.00000	−4.45720	0.24038
	11 × 11	2.42292	7.59883	5.19628	0.00000	−4.45512	0.24018
	13 × 13	2.42287	7.59870	5.19947	0.00000	0.45473	0.24015
	15 × 15	2.42287	7.59889	5.19767	0.00000	−4.45475	0.24015
S′S′S′S′	5 × 5	4.35769	8.69039	6.49219	0.00000	0.00000	0.22331
	7 × 7	4.63582	9.44635	6.92984	0.00000	0.00000	0.20149
	9 × 9	4.65758	9.48765	6.98650	0.00000	0.00000	0.20029
	11 × 11	4.66104	9.49167	6.98560	0.00000	0.00000	0.20012
	13 × 13	4.66150	9.49218	6.98985	0.00000	0.00000	0.20008
	15 × 15	4.66153	9.49243	6.98815	0.00000	0.00000	0.20008
SFSF	5 × 5	6.49199	10.3545	5.12423	−0.00336	0.00000	0.15462
	7 × 7	6.65321	10.8599	5.32248	0.00310	0.00000	0.13358
	9 × 9	6.65472	10.8661	5.34403	0.00037	0.00000	0.13333
	11 × 11	6.65469	10.8649	5.33675	0.00009	0.00000	0.13343
	13 × 13	6.65470	10.8648	5.34013	0.00005	0.00000	0.13342
	15 × 15	6.65470	10.8650	5.33835	0.00002	0.00000	0.13342

(Fig. 1*b*, $K = 3.0$, $v = 0.3$, $h/a = 0.1$, $v/b = 0.5$).
(1) $Q_0 a^3 \times 10^{-3}/D$, (2) $Q_0 a \times 10^{-2}$, (3) Q_0.
[a] Grid points in each element.

the shear deformation (corresponding to the increase of the relative thickness ratio h/a from 0.01 to 0.20) exhibits a tendency of enlargement of the deflection and reduction of the moments and shear forces for plates with elastic foundations. And only when the value of non-dimensional elastic foundation parameter K is big enough ($K > 1.0$), the effects of the shear deformation on the moments and shear forces become significant. The numerical results at several selected locations of the plate for other boundary conditions such as the CCCC, S′S′S′S′ and SFSF under a patch load are tabulated in Tables 6–8. The general trends of the variations of the normalized deflections, moments and shear forces with the dimensions of loading area for CCCC, S′S′S′S′ and SFSF plates are similar to the SSSS plate. However, it is interesting to note that for the SFSF plate, the bending moments at the central point of plate increase first then decrease as the dimensions of loading area, $u \times v$, increase from $0.2a \times 0.2b$ to $a \times b$. The effects of the shear deformation (corresponding to the increase of the relative thickness ratio h/a from 0.01 to 0.20) also exhibit a tendency toward enlargement of the deflection and reduction of the moments and shear forces. However, for values of certain elastic foundation modulus K (e.g. for the CCCC with $K = 1.0$ and S′S′S′S′ plates with $K = 1.0$, 3.0), the shear deformation has the effect to increase the moments and shear forces. This observation is in complete agreement with Kobayashi and Sonoda (1989) for uniformly loaded rectangular plate on Winkler foundation. Tables 9–11 present the

Table 4
Comparison studies of the DQEM solutions with exact solutions for the rectangular plates resting on Winkler foundations under different loading and boundary conditions[a]

K	b/a	Loading	Boundary conditions	v/b	h/a	W_c	$\bar{M}_{xc}$	$\bar{M}_{yc}$	$\bar{Q}_{xm}$	$\bar{M}_{xyl}$
		Patch load								
0.0	1.0	($u=v$)	SSSS	0.2	0.01	0.43493	0.84964	0.84964	0.01667	−0.23875
					Exact[b]	0.43455	0.84697	0.84697	—	−0.23875
				0.5	0.01	2.13348	2.94360	2.94360	0.10196	−1.33495
					Exact[b]	2.13219	2.94504	2.94504	—	−1.33495
				0.8	0.01	3.70586	4.46438	4.46438	0.23808	−2.72893
					Exact[b]	3.70389	4.46731	4.46731	—	−2.72893
		Line load								
	2.0	(u/a = 0.5)	SSSS	0.2	0.01	6.18047	11.0098	6.686762	0.18611	−1.22788
					Exact[b]	6.17553	10.4554	6.63869	—	−1.22792
				0.5	0.01	12.6333	17.9529	8.79775	0.36424	−3.36407
					Exact[b]	12.6274	17.4849	8.69196	—	−3.36401
				0.8	0.01	15.7422	20.8258	8.90832	0.43408	−5.63844
					Exact[b]	15.7353	20.3174	8.73863	—	−5.63853
		Pressure								
3.0	1.0	(u/a = 1.0)	SCSC	1.0	0.10	1.97589	2.27164	2.94078	0.23064	−0.77942
					Exact[c]	1.976	2.272	2.941	0.231	−0.787
			SFSF	1.0	0.10	7.07473	6.29951	1.36490	0.29129	0.00198
					Exact[c]	7.075	6.300	1.366	0.291	—
			SS′SS′	1.0	0.10	3.60326	3.95818	3.90942	0.29615	−0.00134
					Exact[c]	3.603	3.957	3.906	0.296	—

[a] For pach loading: $W_c = w_c \quad D/(10^{-3} \times qa^4)$; $\bar{M}_{xc} = M_{xc}/(10^{-2} \times qa^2)$; $\bar{M}_{yc}/(10^{-2} \times qa^2)$; $\bar{Q}_{xm} = Q_{xm}/(qa)$; $\bar{M}_{xyl} = M_{xyl}/(10^{-2} \times qa^2)$; where w_c, and M_{yc} are the defection, bending moments at the plate center $x = 0.5a$, $y = 0.5b$; Q_{xm} and M_{xyl} are the shear force Q_x at the mid-side of $x = 0$ and the twisting moment at the plate corner $x = 0$, $y = 0$.
For concentrated loading: $W_c = w_c D/(10^{-3} \times Q_0 a^3)$; $\bar{M}_{xc} = M_{xc}/(10^{-2} \times Q_0 a)$; $\bar{M}_{yc} = M_{yc}/(10^{-2} \times Q_0 a)$; $\bar{Q}_{xm} = Q_{xm}/Q_0$; $\bar{M}_{xyl} = M_{xyl}/(10^{-2} \times Q_0 a)$.
[b] Exact solution obtained by using theoretical formulas based on the classical thin plate theory (Pilkey, 1994).
[c] Exact solution obtained by Kobayashi and Sonoda (1989) based on the Mindlin plate theory.

numerical results of a square plate subjected to a concentrated line load along the central line $x = 0.5$ under SSSS, CCCC and SFSF boundary conditions respectively. It is evident from these tables that as the length of the loading line, v, increases from $0.2b$ to b, the central deflection, moment M_x, and shear forces increase for all these boundary conditions. The moment M_y at the center of plate, however, shows a different way of variation. That is, for some values of foundation modulus K and some relative thickness ratio h/a (for SSSS and CCCC plates, when $K = 1.0$ or 3.0 and $h/a = 0.01$, 0.20; and for SFSF plate, when $K = 1.0$ and $h/a = 0.01$), the value of M_y increases with v, whereas for some other values of K and h/a, it increases first and then decreases with the increasing value of v. As the relative thickness ratio h/a increases from 0.01 to 0.20, the central deflections for all the SSSS, CCCC and SFSF plates increase. When $K = 1.0$, the moment M_x and shear forces for the SSSS and SFSF plates demonstrate very slight change in values, the moment M_y for SSSS plate also shows a minimum variation but for the SFSF plate, M_y decreases with the increase of h/a. When $K = 3.0$ and 5.0, all the moments and shear forces of the SSSS and SFSF plates decrease as the relative thickness ratio h/a increases from 0.01 to 0.20. For the CCCC plate, the trends of variations of the moments and shear forces are very similar to those of plate subjected to a patch load. The effects of the elastic foundation modulus K on the deflections, moments and shear forces of all the SSSS, CCCC and SFSF plates under

Table 5
Numerical results for a SSSS square plate subjected to a patch load and resting on Winkler foundation

h/a	$u=v$	$w^{(1)}$ $x/a=0.5$ $y/b=0.5$	$M_x^{(2)}$ $x/a=0.5$ $y/b=0.5$	$M_y^{(2)}$ $x/a=0.5$ $y/b=0.5$	$M_{xy}^{(2)}$ $x/a=0.0$ $y/b=0.0$	$Q_x^{(3)}$ $x/a=0.0$ $y/b=0.5$	$Q_{xy}^{(3)}$ $x/a=0.5$ $y/b=0.0$
K = 1.0							
0.01	0.2	0.43391	0.84832	0.84832	−0.23806	0.01662	0.01662
	0.5	2.12814	2.93673	2.93673	−1.33128	0.10163	0.10163
	0.8	3.69622	4.45605	4.45605	−2.72231	0.23743	0.23743
	1.0	4.05381	4.77504	4.77504	−3.23825	0.33729	0.33729
0.20	0.2	0.58241	0.84800	0.84800	−0.23790	0.01661	0.01661
	0.5	2.64172	2.93515	2.93515	−1.33047	0.10156	0.10156
	0.8	4.47511	4.45332	4.45332	−2.72089	0.23731	0.23731
	1.0	4.88837	4.77203	4.77203	−3.23906	0.33686	0.33686
K = 3.0							
0.01	0.2	0.36624	0.76019	0.76019	−0.19190	0.01252	0.01252
	0.5	1.77461	2.48110	2.48110	−1.08839	0.07991	0.07991
	0.8	3.05800	3.64019	3.64019	−2.28020	0.19770	0.19770
	1.0	3.34854	3.87501	3.87501	−2.74863	0.29323	0.29323
0.20	0.2	0.48386	0.74226	0.74226	−0.18383	0.01184	0.01184
	0.5	2.13004	2.39570	2.39570	−1.04502	0.07614	0.07614
	0.8	3.55525	3.49464	3.49464	−2.19883	0.19037	0.19037
	1.0	3.87274	3.71598	3.71598	−2.65989	0.28463	0.28463
K = 5.0							
0.01	0.2	0.18821	0.52321	0.52321	−0.07264	0.00207	0.00207
	0.5	0.84766	1.27788	1.27788	−0.45614	0.02371	0.02371
	0.8	1.38961	1.51698	1.51698	−1.11909	0.09317	0.09317
	1.0	1.50610	1.54013	1.54013	−1.45953	0.17685	0.17685
0.20	0.2	0.25270	0.48459	0.48459	−0.06237	0.00147	0.00147
	0.5	0.94597	1.13359	1.13359	−0.39540	0.01911	0.01911
	0.8	1.44778	1.31402	1.31402	−0.98973	0.08137	0.08137
	1.0	1.55069	1.32770	1.32770	−1.31092	1.6203	0.16203

(v = 0.3; b/a = 1.0).
(1) $qa^4 \times 10^{-3}/D$; (2) $qa^2 \times 10^{-2}$; (3) qa.

the concentrated line loading conditions are also similar to those under the patch loading conditions. All the values of these parameters decrease with the increase of K since the elastic foundation becomes stiffer.

4. Concluding remarks

The first known two-dimensional differential quadrature element method (DQEM) has been developed for the static analysis of rectangular thick plates on Winkler foundations based on the first-order shear deformation theory. The approach developed here is a combination of the differential quadrature technique and the domain decomposition method. The reliability of the DQEM solutions for the title problem has been examined by the convergence and comparison studies. Very close agreement has been achieved between present solutions and those obtained using analytical or other methods. The detailed parametric studies have been carried out for the rectangular plates with different boundary conditions on Winkler foundations and subjected to a patch load or a concentrated line load. The relations between the numerical results for the deflection, moments and shear forces, and the dimensions of loading area/line, relative thickness ratio and elastic foundation modulus have been well revealed by all

Table 6
Numerical results for a CCCC square plate subjected to a patch load and resting on Winkler foundation

h/a	$u=v$	$w^{(1)}$ $x/a=0.5$ $y/b=0.5$	$M_x^{(2)}$ $x/a=0.5$ $y/b=0.5$	$M_y^{(2)}$ $x/a=0.5$ $y/b=0.5$	$M_{xy}^{(2)}$ $x/a=0.0$ $y/b=0.0$	$Q_x^{(3)}$ $x/a=0.0$ $y/b=0.5$	$Q_{xy}^{(3)}$ $x/a=0.5$ $y/b=0.0$
K = 1.0							
0.01	0.2	0.20083	0.63955	0.63955	0.00000	0.03093	0.03093
	0.5	0.84922	1.79196	1.79196	0.00000	0.17068	0.17068
	0.8	1.23171	2.25508	2.25508	0.00000	0.33738	0.33738
	1.0	1.26682	2.28865	2.28865	0.00000	0.43873	0.43873
0.20	0.2	0.35645	0.64648	0.64648	0.00000	0.02198	0.02198
	0.5	1.40043	1.82878	1.82878	0.00000	0.12896	0.12896
	0.8	2.07494	2.31724	2.31724	0.00000	0.28072	0.28072
	1.0	2.16894	2.35299	2.35299	0.00000	0.38175	0.38175
K = 3.0							
0.01	0.2	0.19040	0.61726	0.61726	0.00000	0.02856	0.02856
	0.5	0.80034	1.68997	1.68997	0.00000	0.15932	0.15932
	0.8	1.15706	2.10163	2.10163	0.00000	0.31969	0.31969
	1.0	1.18962	2.13024	2.13024	0.00000	0.42031	0.42031
0.20	0.2	0.32825	0.60948	0.60948	0.00000	0.01899	0.01899
	0.5	1.26521	1.65830	1.65830	0.00000	0.11374	0.11374
	0.8	1.85698	2.04976	2.04976	0.00000	0.25475	0.25475
	1.0	1.93792	2.07103	2.07103	0.00000	0.35378	0.35378
K = 5.0							
0.01	0.2	0.14237	0.51347	0.51347	0.00000	0.01778	0.01778
	0.5	0.57603	1.22110	1.22110	0.00000	0.10733	0.10733
	0.8	0.81519	1.40227	1.40227	0.00000	0.23819	0.23819
	1.0	0.83615	1.40901	1.40901	0.00000	0.33540	0.33540
0.20	0.2	0.22316	0.46694	0.46694	0.00000	0.00844	0.00844
	0.5	0.76897	1.02834	1.02834	0.00000	0.05894	0.05894
	0.8	1.06599	1.08969	1.08969	0.00000	0.15871	0.15871
	1.0	1.10136	1.06478	1.06478	0.00000	0.24942	0.24942

(v = 0.3; b/a = 1.0).
(1) $qa^4 \times 10^{-3}/D$; (2) $qa^2 \times 10^{-2}$; (3) qa.

Table 7
Numerical results for a S′S′S′S′ square plate subjected to a patch load and resting on Winkler foundation

h/a	$u = v$	$w^{(1)}$ $x/a = 0.5$ $y/b = 0.5$	$M_x^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_y^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_{xy}^{(2)}$ $x/a = 0.0$ $y/b = 0.0$	$Q_x^{(3)}$ $x/a = 0.0$ $y/b = 0.5$	$Q_y^{(3)}$ $x/a = 0.5$ $y/b = 0.0$
$K = 1.0$							
0.01	0.2	0.43150	0.84298	0.84298	0.00000	0.02549	0.02549
	0.5	2.12971	2.93571	2.93571	0.00000	0.14754	0.14754
	0.8	3.72560	4.48211	4.48211	0.00000	0.31432	0.31432
	1.0	4.10868	4.81613	4.81613	0.00000	0.40101	0.40101
0.20	0.2	0.63603	0.89632	0.89632	0.00000	0.02341	0.02341
	0.5	2.93557	3.19962	3.19962	0.00000	0.13833	0.13833
	0.8	5.03917	4.95894	4.95894	0.00000	0.30551	0.30551
	1.0	5.52470	5.34162	5.34162	0.00000	0.41260	0.41260
$K = 3.0$							
0.01	0.2	0.36452	0.75614	0.75614	0.00000	0.01996	0.01996
	0.5	1.77552	2.47981	2.47981	0.00000	0.11812	0.11812
	0.8	3.07785	3.65609	3.65609	0.00000	0.26038	0.26038
	1.0	3.38572	3.89860	3.89860	0.00000	0.34402	0.34402
0.20	0.2	0.51581	0.76993	0.76993	0.00000	0.01660	0.01660
	0.5	2.30699	2.54861	2.54861	0.00000	0.10198	0.10198
	0.8	3.89942	3.79011	3.79011	0.00000	0.23828	0.23828
	1.0	4.26265	4.04989	4.04989	0.00000	0.33786	0.33786
$K = 5.0$							
0.01	0.2	0.18777	0.52185	0.52185	0.00000	0.00569	0.00569
	0.5	0.84768	1.27690	1.27690	0.00000	0.04175	0.04175
	0.8	1.39336	1.51762	1.51762	0.00000	0.11952	0.11952
	1.0	1.51325	1.53869	1.53869	0.00000	0.23382	0.23382
0.20	0.2	0.25680	0.48722	0.48722	0.00000	0.00281	0.00281
	0.5	0.97022	1.14891	1.14891	0.00000	0.02663	0.02663
	0.8	1.49861	1.34456	1.34456	0.00000	0.09517	0.09517
	1.0	1.60959	1.36234	1.36234	0.00000	0.17717	0.17717

($v = 0.3$; $b/a = 1.0$).
(1) $qa^4 \times 10^{-3}/D$; (2) $qa^2 \times 10^{-2}$; (3) qa.

Table 8
Numerical results for a SFSF square plate subjected to a patch load and resting on Winkler foundation

h/a	$u = v$	$w^{(1)}$ $x/a = 0.5$ $y/b = 0.5$	$M_x^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_y^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_{xy}^{(2)}$ $x/a = 0.0$ $y/b = 0.0$	$Q_x^{(3)}$ $x/a = 0.0$ $y/b = 0.5$	$Q_y^{(3)}$ $x/a = 0.5$ $y/b = 0.0$
$K = 1.0$							
0.01	0.2	0.88771	1.22419	0.74085	0.00000	0.02309	0.00000
	0.5	4.84432	5.18508	2.29314	0.00000	0.14032	0.00000
	0.8	10.0239	9.68240	2.95652	0.00000	0.32775	0.00000
	1.0	12.9591	12.1067	2.62639	0.00000	0.46058	0.00000
0.20	0.2	1.05995	1.22346	0.73768	0.00000	0.02298	0.00000
	0.5	5.50703	5.18330	2.25719	0.00000	0.13942	0.00000
	0.8	11.1810	9.69039	2.79913	0.00000	0.32428	0.00000
	1.0	14.3658	12.1425	2.34449	0.00000	0.45732	0.00000
$K = 3.0$							
0.01	0.2	0.52930	0.87430	0.65054	0.00000	0.01261	0.00000
	0.5	2.78559	3.18855	1.80543	0.00000	0.08060	0.00000
	0.8	5.54332	5.36922	1.98293	0.00000	0.19966	0.00000
	1.0	6.98532	6.38743	1.42945	0.00000	0.29371	0.00000
0.20	0.2	0.63604	0.84954	0.64501	0.00000	0.01205	0.00000
	0.5	3.07778	3.05869	1.77171	0.00000	0.07697	0.00000
	0.8	5.90618	5.11354	1.87114	0.00000	0.19052	0.00000
	1.0	7.34272	6.07408	1.22369	0.00000	0.28215	0.00000
$K = 5.0$							
0.01	0.2	0.19103	0.51990	0.50244	0.00000	0.00148	0.00000
	0.5	0.87402	1.24900	1.09725	0.00000	0.01860	0.00000
	0.8	1.48569	1.40170	0.85981	0.00000	0.07642	0.00000
	1.0	1.67925	1.31949	0.33852	0.00000	0.14175	0.00000
0.20	0.2	0.25596	0.48550	0.47423	0.00000	0.00122	0.00000
	0.5	0.97002	1.13063	1.01519	0.00000	0.01610	0.00000
	0.8	1.52072	1.26470	0.79417	0.00000	0.06768	0.00000
	1.0	1.67130	1.20032	0.28289	0.00000	0.13389	0.00000

($v = 0.3$; $b/a = 1.0$).
(1) $qa^4 \times 10^{-3}/D$; (2) $qa^2 \times 10^{-2}$; (3) qa.

Table 9
Numerical results for a SSSS square plate subjected to a concentrated line load along $x/a = 0.5$ and resting on Winkler foundation

h/a	v	$w^{(1)}$ $x/a=0.5$ $y/b=0.5$	$M_x^{(2)}$ $x/a=0.5$ $y/b=0.5$	$M_y^{(2)}$ $x/a=0.5$ $y/b=0.5$	$M_{xy}^{(2)}$ $x/a=0.0$ $y/b=0.0$	$M_x^{(2)}$ $x/a=0.0$ $y/b=0.5$	$Q_x^{(3)}$ $x/a=0.0$ $y/b=0.5$
$K = 1.0$							
0.01	0.2	2.23724	6.09667	5.00118	−1.20252	0.00000	0.08107
	0.5	4.91491	10.3835	7.87894	−2.82297	0.00000	0.18134
	0.8	6.43463	12.3652	8.99956	−3.90104	0.00000	0.23803
	1.0	6.72902	12.7334	9.19212	−4.12824	0.00000	0.24879
0.20	0.2	3.20900	6.09520	5.00065	−1.20148	0.00000	0.08123
	0.5	6.51277	10.3781	7.87166	−2.82109	0.00000	0.18116
	0.8	8.30303	12.3572	8.98577	−3.89859	0.00000	0.23789
	1.0	8.64523	12.71343	9.18757	−4.12776	0.00000	0.24878
$K = 3.0$							
0.01	0.2	1.89315	5.64756	4.55269	−0.96805	0.00000	0.06029
	0.5	4.12848	9.36044	6.86234	−2.28521	0.00000	0.13393
	0.8	5.37807	10.9946	7.64328	−3.17605	0.00000	0.17446
	1.0	5.61838	11.2934	7.76858	−3.36551	0.00000	0.18199
0.20	0.2	2.70737	5.55412	4.46152	−0.92688	0.00000	0.05713
	0.5	5.37045	9.15332	6.66265	−2.19047	0.00000	0.12626
	0.8	6.77243	10.7227	7.38352	−3.04724	0.00000	0.16440
	1.0	7.03718	10.9977	7.50809	−3.23180	0.00000	0.17158
$K = 5.0$							
0.01	0.2	0.98758	4.43561	3.34474	−0.36263	0.00000	0.00762
	0.5	2.06236	6.61744	4.16078	−0.89207	0.00000	0.01436
	0.8	2.60683	7.33966	4.08047	−1.29159	0.00000	0.01485
	1.0	2.70637	7.45837	4.03867	−1.38137	0.00000	0.01447
0.20	0.2	1.52789	4.22137	3.14104	−0.31085	0.00000	0.00513
	0.5	2.70561	6.18777	3.79375	−0.76777	0.00000	0.00871
	0.8	3.22352	6.81291	3.66873	−1.11543	0.00000	0.00812
	1.0	3.31355	6.90365	3.63325	−1.19591	0.00000	0.00771

($v = 0.3$; $b/a = 1.0$).
(1) $Q_0a^3 \times 10^{-3}/D$; (2) $Q_0a \times 10^{-2}$; (3) Q_0.

Table 10
Numerical results for a CCCC square plate subjected to a concentrated line load along $x/a = 0.5$ and resting on Winkler foundation

h/a	v	$w^{(1)}$ $x/a = 0.5$ $y/b = 0.5$	$M_x^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_y^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_{xy}^{(2)}$ $x/a = 0.0$ $y/b = 0.0$	$M_x^{(2)}$ $x/a = 0.0$ $y/b = 0.5$	$Q_x^{(3)}$ $x/a = 0.0$ $y/b = 0.5$
$K = 1.0$							
0.01	0.2	1.05822	5.03622	3.94987	0.00000	−2.42848	0.15100
	0.5	2.14941	7.84246	5.46544	0.00000	−5.13392	0.31077
	0.8	2.57504	8.71005	5.73384	0.00000	−6.23010	0.36925
	1.0	2.61140	8.79803	5.75596	0.00000	−6.32161	0.37304
0.20	0.2	2.06605	5.07437	3.98285	0.00000	−2.08470	0.10733
	0.5	3.83216	7.96667	5.50158	0.00000	−4.53224	0.22876
	0.8	4.56574	8.95973	5.71394	0.00000	−5.70430	0.28178
	1.0	4.66111	9.07654	5.70866	0.00000	−5.86344	0.28800
$K = 3.0$							
0.01	0.2	1.00458	4.92098	3.83506	0.00000	−2.25473	0.13889
	0.5	2.03329	5.59470	5.22160	0.00000	−4.75794	0.28465
	0.8	2.43132	8.40467	5.43589	0.00000	−5.76511	0.33706
	1.0	2.46515	8.48744	5.45327	0.00000	−5.84864	0.34037
0.20	0.2	1.92123	4.88181	3.79209	0.00000	−1.84180	0.09213
	0.5	3.51534	7.55043	5.09920	0.00000	−3.99628	0.19541
	0.8	4.16303	8.43419	5.21533	0.00000	−5.01964	0.23940
	1.0	4.24595	8.53535	5.19720	0.00000	−5.15715	0.24434
$K = 5.0$							
0.01	0.2	0.75764	4.38295	3.29987	0.00000	−1.46428	0.08402
	0.5	1.49945	6.44293	4.09568	0.00000	−3.05032	0.16682
	0.8	1.77129	6.98913	4.06920	0.00000	−3.65626	0.19224
	1.0	1.79364	7.04820	4.06589	0.00000	−3.70403	0.19347
0.20	0.2	1.37986	4.13184	3.05360	0.00000	−0.97093	0.03908
	0.5	2.34167	5.95560	3.59073	0.00000	−2.08024	0.07968
	0.8	2.68068	6.44159	3.38840	0.00000	−2.57931	0.09332
	1.0	2.71965	6.48711	3.33157	0.00000	−2.64156	0.09408

($v = 0.3$; $b/a = 1.0$).
(1) $Q_0 a^3 \times 10^{-3}/D$; (2) $Q_0 a \times 10^{-2}$; (3) Q_0.

Table 11
Numerical results for a SFSF square plate subjected to a concentrated line load along $x/a = 0.5$ and resting on Winkler foundation

h/a	ν	$w^{(1)}$ $x/a = 0.5$ $y/b = 0.5$	$M_x^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_y^{(2)}$ $x/a = 0.5$ $y/b = 0.5$	$M_{xy}^{(2)}$ $x/a = 0.0$ $y/b = 0.0$	$M_x^{(2)}$ $x/a = 0.0$ $y/b = 0.5$	$Q_x^{(3)}$ $x/a = 0.0$ $y/b = 0.5$
$K = 1.0$							
0.01	0.2	4.54410	8.00778	4.45463	0.00001	0.00000	0.11394
	0.5	10.9502	15.3878	6.44371	0.00001	0.00000	0.26699
	0.8	16.8939	21.0549	6.49059	0.00000	0.00000	0.38540
	1.0	20.7363	24.3507	5.71181	0.00156	0.00000	0.44090
0.20	0.2	5.63654	8.00420	4.43968	0.00028	0.00000	0.11363
	0.5	12.8797	15.3818	6.36102	0.00010	0.00000	0.26492
	0.8	19.3891	21.0644	6.22179	0.00014	0.00000	0.37393
	1.0	23.5561	24.4297	5.27192	0.00081	0.00000	0.43456
$K = 3.0$							
0.01	0.2	2.72208	6.22805	3.99503	0.00001	0.00000	0.06074
	0.5	6.37567	10.9413	5.35594	0.00001	0.00000	0.13517
	0.8	9.49088	13.9056	4.87395	0.00000	0.00000	0.17605
	1.0	11.3488	15.3355	3.82251	0.00156	0.00000	0.18147
0.20	0.2	3.48097	6.09983	3.96698	0.00028	0.00000	0.05817
	0.5	7.47746	10.6351	5.27209	0.00010	0.00000	0.12779
	0.8	10.6651	13.4504	4.67108	0.00014	0.00000	0.16303
	1.0	12.5101	14.8327	3.49231	0.00081	0.00000	0.16470
$K = 5.0$							
0.01	0.2	1.00193	4.41904	3.23893	0.00001	0.00000	0.00464
	0.5	2.12190	6.55991	3.75469	0.00001	0.00000	0.00289
	0.8	2.77172	7.19087	2.96542	0.00000	0.00000	−0.01662
	1.0	2.99234	7.17869	2.05126	0.00156	0.00000	−0.04025
0.20	0.2	1.54449	4.22625	3.08827	0.00028	0.00000	0.00386
	0.5	2.76003	6.18671	3.52741	0.00010	0.00000	0.00187
	0.8	3.35012	6.76680	2.78434	0.00014	0.00000	−0.01515
	1.0	3.51724	6.79381	1.90160	0.00081	0.00000	−0.03730

($\nu = 0.3$; $b/a = 1.0$).
(1) $Q_0 a^3 \times 10^{-3}/D$; (2) $Q_0 a \times 10^{-2}$; (3) Q_0.

these solution data. It has been demonstrated in this paper that the DQEM is simple in numerical implementation, accurate in solution and more flexible than the global DQ method. Therefore it is a very powerful solution tool for the problems of thick plates with discontinuities.

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